Practical Guidelines for PD and PI Delay-Based Controllers.

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Abstract: This paper focuses on the control scheme design of two different control schemes using delays. These two low-complexity controllers are direct alternatives for the PD and PI low-order controllers. More precisely, first, we study a PD controller using the Euler approach for approximating the derivative action. Second, we analyze the implications of imposing a delay in the error signal on the integral action of the PI controller for closed-loop response manipulation purposes. Our main contribution lies in proposing some practical guidelines for the tuning of these delayed control schemes such that the closed-loop system is stable. To this end, the criteria developed in this work makes use of the well-known $\mathcal{D}$-partition curves method avoiding crossing direction analysis. Finally, in order to test the effectiveness of the proposed methodology, some numerical examples are presented.

Keywords: LTI Systems, Delay Control, Stability Crossing Curves, PID controllers, SISO systems.

1. INTRODUCTION

It is well recognized that low-order controllers are one of the most widely applied strategies to control industrial processes (see, e.g., Aström and Hägglund (2001); O'Dwyer (2009)). Such a "popularity" is mainly due to their particular distinct features: simplicity and ease of implementation. Among these controllers, those of PID-type are known to be able to cope with uncertainties, disturbances, elimination of steady-state errors and transient response improvement (Aström and Hägglund (1995); Méndez-Barrios et al. (2008); Ramírez et al. (2016)). However, the main drawbacks of PID controllers, reported in Aström and Hägglund (1995), lie in the tuning of the derivative term, which may amplify high-frequency measurement noise. In fact, as mentioned in Aström and Hägglund (2001); O'Dwyer (2009) the above arguments advise to avoid the derivative action in most applications.

In order to circumvent such a problem, the Euler approximation of the derivative:

$$y'(t) \approx \frac{y(t) - y(t-\epsilon)}{\epsilon}, \tag{1}$$

for small $\epsilon > 0$, seems to be the simplest way to replace the derivative action by using its delay-difference approximation counterpart (Niculescu and Michiels (2004)). However, it is important to point out that the presence of a delay in the feedback loop of continuous-time systems is accompanied among others with oscillations, instability and bandwidth sensitivity (see, for instance, Niculescu (2001); Michiels and Niculescu (2014)). It is also worth to mention that there exist some situations when the delay may induce stability, as explained in the classical example shown in Abdallah et al. (1993), where an oscillator is controlled by one delay "block": (gain, delay), with positive gains and small delay values (a detailed analysis of such an approach can be found in Niculescu et al. (2010)). In addition, it has also been reported that there exist situations where an appropriate selection of the delay parameter may improve the system's response (see, for instance, Chen (1987)). Inspired by the above observations, the design of low-order controllers with delay as a *control parameter* have been addressed in several works, for example, Niculescu and Michiels (2004) (stabilizing chains of integrators by using delays), Kharitonov et al. (2005) (multiple delay blocks), Mazenc et al. (2003) (bounded input,single delay), to mention a few.

In this paper, we present two alternatives using *delays* as *design parameters* to the classical PD and PI control schemes. On one hand, we study the $P\delta$ controller, which consists in substituting directly the derivative part of a PD controller by the above mentioned Euler approximation. One of the main benefits in considering such an approximation is that most control schemes are implemented digitally. As a consequence, a numerical method needs to be considered in order to achieve a derivative action. In this vein, one of the main features of the $P\delta$ controller is that is easier to implement on such platforms and its model approximates more accurately the derivative action alike applied for small delay values. On the other hand, based on a PI controller, we consider a delay in the error signal only in the integral action. This provides an extra degree of freedom in the tuning of this controller maintaining the most important feature of the PI controller, which is the null steady state error in the regulation of zero type systems (open-loop systems with no poles at the origin).

2. PROBLEM FORMULATION

Consider the class of proper SISO open-loop systems given by the transfer function:

$$G(s) := \frac{P(s)}{Q(s)} = \boldsymbol{C}^T(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}, \quad (2)$$

where $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}^T)$ is the state-space representation of the open-loop system, with P and Q polynomials of degree m and n, respectively. Whose highest degree coefficients are denoted by $p_m \neq 0$ and $p_n \neq 0$. In the remaining part of the paper, we will consider that the following assumption is satisfied.

Assumption 1. Polynomials P and Q satisfy the following conditions:

(i) $\deg Q > \deg P$.
(ii) $P(s)$ and $Q(s)$ are coprime polynomials.

It is clear that Assumption 1-(i) states that the system is causal. If Assumption 1-(ii) is not fulfilled, this implies that there exist a non constant common factor $c(s)$, such that $P(s) = c(s)\tilde{P}(s)$ and $Q(s) = c(s)\tilde{Q}(s)$. In such a case, choosing $c(s)$ to be of the highest possible degree, the analysis can be pursued if $c(s)$ is a Hurwitz polynomial, otherwise, the system will remain unstable independently of the control action.

The problems considered in this paper can be summarized as follows:

Problem 1. Find explicit conditions on the parameters $(\tau, k_p, k_\delta) \in \mathbb{R}_+ \times \mathbb{R}^2$, such that the $P\delta$ controller:

$$C_\delta(s) = k_p + k_\delta \frac{1 - e^{-\tau s}}{\tau}, \quad (3)$$

asymptotically stabilizes the closed-loop system.

Problem 2. Find explicit conditions on the parameters $(\tau, k_p, k_i) \in \mathbb{R}_+ \times \mathbb{R}^2$, such that the $P\delta I$ controller:

$$C_i(s) = k_p + k_i \frac{e^{-\tau s}}{s}, \quad (4)$$

asymptotically stabilizes the closed-loop system.

In the following, we consider the vectors $\boldsymbol{k^\delta} := [k_p, k_\delta]^T$ and $\boldsymbol{k^i} := [k_p, k_i]^T$ referring to the $P\delta$ and $P\delta I$ controllers, respectively. The real functions $\Re(\sigma, \omega)$ (and $\Im(\sigma, \omega)$) stand for the real (and imaginary part) of $G^{-1}(\sigma + i\omega)$. Moreover, from a geometric point of view, for a fixed $\tau^* \in \mathbb{R}_+$, we can define the collection of all controller gains $\boldsymbol{k^\delta} \in \mathbb{R}^2$ as points in the k_p-k_δ parameters plane. Therefore, Problem 1 can be stated as the task of finding at least one region in the k_p-k_δ parameters-plane such that, for all $\boldsymbol{k^\delta}-$points inside this region, the characteristic equation of the closed-loop system *has all* of its roots in the LHP (left-half plane) of the complex plane. A region of the k_p-k_δ parameters-plane with such a feature is defined as a *stability region*. Without any loss of generality, the same can be stated for the controller $P\delta I$.

3. MOTIVATING EXAMPLES

In this section we depict two motivating examples of the use of each controller ($P\delta$ and $PI\delta$). The main purpose of these is to enhance some advantages regarding the stability of the closed-loop system with respect to their low-order controllers counterparts (PD and PI).

Example 1. Consider the following open-loop transfer function:

$$G(s) = \frac{1}{s^3 - s^2 + 4s - 6}, \quad (5)$$

with two stable poles $s_{1,2} = -0.17 \pm 2.1i$ and a real unstable one $s_3 = 1.34$. Considering the use of the well known PD controller, such case leads to the following characteristic equation:

$$\Delta(s) = s^3 - s^2 + (k_d + 4)s + (k_p - 6) = 0. \quad (6)$$

Using the Routh-Hurwitz stability criterion is easy to prove that a necessary condition for closed-loop stability lies in having a positive second order term. Notice that in this case, the PD controller does not have the necessary impact on the characteristic equation to achieve it. In fact, it is only possible to design the zero and first order terms through this control scheme.

In contrast, using the MatLab package DDE-BIFTOOL we compute the location of the rightmost roots of the characteristic equation of the closed-loop system, now by tunning a $P\delta$ controller with parameters $k^\delta = [6.4, -3.4]^T$ with a fixed delay $\tau = 1s$. As depicted in Fig. 1, all of these are located inside the LHP, therefore, the system can be stabilized with such controller.

Example 2. Consider the following open-loop transfer function:

$$G(s) = \frac{1}{s^2 - 0.1s - 0.02}, \quad (7)$$

with poles $s_1 = 0.2$ and $s_2 = -0.1$. Considering the use of the PI controller, such a case leads to the following characteristic equation:

$$\Delta(s) = s^3 - 0.1s^2 + (k_p - 0.02)s + k_i = 0. \quad (8)$$

In a similar fashion that the first example, using the Routh-Hurwitz stability criterion it arises the necessary condition of having only positive terms in this polynomial in order to achieve stability. Also for this example, the use of the PI controller is not enough for this purpose due to its null impact on the second order negative term. Now, we compute the location of the rightmost roots of the characteristic equation by considering a $P\delta I$ controller with parameters $k^i = [80, 200]^T$ with a fixed delay $\tau = 0.3s$. These results are shown in Fig. 1, since the roots are located inside the LHP, therefore, the addition of the delayed action gives the possibility of achieving stability.

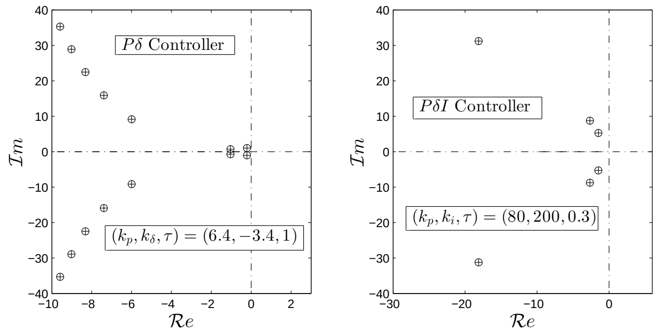

Fig. 1. Motivating examples – Closed-loop system rightmost roots location.

4. CONTROL SCHEME DESIGN

In this section the main results proposed in this paper are presented and derived. First, the tuning methodology is explained in detail. Second, we describe some insights of the $P\delta$ controller and we give conditions of the parameters (k_p, k_δ, τ) such that the characteristic equation has at least one root in $s = \sigma + i\omega$. Third, the same is properly proposed for the $P\delta I$ controller. Finally, we characterize the delay interval such that the closed-loop system remains stable given a stabilizing triplet (k_p, k_i, τ).

4.1 Delayed Controllers Tuning Methodology

In this section, without any loss of generality we consider both controllers ($C_\delta(s)$ and $C_i(s)$) as $C(s, e^{-\tau s})$ with its respective gains $\boldsymbol{k} \in \mathbb{R}^2$ ($\boldsymbol{k}_\delta$ and $\boldsymbol{k}_i$). The characteristic equation of this closed-loop scheme rewrites as follows:

$$\Delta_\tau(s) = C(s, e^{-\tau s}) + G^{-1}(s) = 0. \tag{9}$$

It is well known that, in order to achieve asymptotic stability, all the roots of (9) have to remain in the LHP of the complex plane.

Remark 1. It is clear that if we take the complex conjugate of (9), the following equality holds: $\Delta_\tau(\sigma + i\omega) = \overline{\Delta_\tau(\sigma - i\omega)}$. Therefore, in the rest of the paper we will consider only nonnegative frequencies ω.

Now, let $\tau \in \mathbb{R}_+$ and $\sigma \in \mathbb{R}_+ \cup \{0\}$ be fixed values, we introduce the following set:

$$\mathcal{T}(\sigma) := \left\{ \boldsymbol{k} \in \mathbb{R}^2 \,|\Delta_\tau(\sigma + i\omega) = 0, \forall \omega \in \Omega \right\}, \tag{10}$$

with $\Omega \subset \mathbb{R}_+$, which is the set of all ω values such that $\Delta_\tau(\sigma + i\omega) = 0$ for a fixed pair (τ, σ). Such set of frequencies is characterized in Propositions 1 and 2 considering the $P\delta$ and $P\delta I$ controller, respectively. Roughly speaking, the set $\mathcal{T}(\sigma)$ contains all gain vectors k such that the characteristic equation of the closed-loop system has at least one root on a vertical line in σ on the complex plane. In other words, Ω includes all the frequencies for which the gains $k \in \mathbb{R}^2$ define some crossing points, that is, points located in the complex plane on the line $\Re\{s\} = \sigma$.

With this notation, it is clear that all possible gain vectors k such that the system has at least one root in the RHP (right-half plane) or in the imaginary axis of the complex plane can be characterized by:

$$\bar{\mathcal{T}}^+ := \bigcup_{\sigma \in \mathbb{R}_+ \cup \{0\}} \mathcal{T}(\sigma). \tag{11}$$

Therefore, all stabilizing controllers k are contained in the set $\bar{\mathcal{T}}^- := \mathbb{R}^2 \setminus \bar{\mathcal{T}}^+$. However, it is worthy to notice that we focus in a particular region of the parameters-space of $k \subset \mathbb{R}^2$ for computational purposes. This process is explained below.

First of all, it is necessary to enhance the importance of the set $\mathcal{T}(0)$. This set contains all possible gain vectors k such that the characteristic equation (9) has at least one root on the imaginary axis. In other words, $\mathcal{T}(0)$ is nothing else that the so-called "stability crossing curves" (see, e.g. Gu et al. (2005), for the definition). Notice that any continuous variation of k such that $k \notin \mathcal{T}(0)$ implies that no roots exchange through the imaginary axis is achieved. Taking into account this argument, it is easy to understand how these *stability crossing curves* partition the parameters-space in regions in which any choice of k implies that (9) has a finite number of roots on the RHP. Second, notice that if some element of $\mathcal{T}(\sigma)$ with $\sigma > 0$ is located inside one of this regions implies that the characteristic equation (9) has at least one unstable root in the RHP. Therefore, this can be labeled as an *unstable region.* Finally, any region which is not unstable is a subset of $\bar{\mathcal{T}}^-$ and can be labeled as a *stability region.*

4.2 Pδ Controller

Consider the use of the $P\delta$ controller shown in (3). The corresponding control law to be applied can be described as:

$$u(t) = k_p e(t) + k_\delta \left(\frac{e(t) - e(t-\tau)}{\tau} \right). \tag{12}$$

Notice that the delayed action resembles the simplest approximation of a derivative given by the Euler approximation (1) previously discussed in the Introduction. Roughly speaking, for small values of τ this controller approximates to a classical PD controller as $C_d(s) = k_p + k_\delta s \approx C_\delta(s)$. In order to study its stability, the characteristic equation of the closed-loop system can be computed by $C_\delta(s)G(s) + 1 = 0$, which straightforwardly leads us to:

$$\Delta_\delta(s) = k_p + k_\delta \left(\frac{1 - e^{-\tau s}}{\tau} \right) + G^{-1}(s) = 0. \tag{13}$$

The following result shown in this section works as a tool for describing the behavior of the roots of this equation.

Proposition 1. Let $\tau \in \mathbb{R}_+$ and $\sigma \in \mathbb{R}$ be fixed values. Then, the characteristic equation (13) has at least one root in $s = \sigma + i\omega$, iff:

$$k_p = -\Re\,(\sigma, \omega) + \left(e^{-\tau\sigma} \csc(\tau\omega) - cot(\tau\omega)\right) \Im\,(\sigma, \omega)\,, \tag{14}$$

$$k_\delta = -\tau e^{\tau\sigma} \csc(\tau\omega) \Im\,(\sigma, \omega)\,. \tag{15}$$

with $\omega \in \Omega_\delta$, where the set Ω_δ is defined by:

$$\Omega_\delta := \left\{ \omega \in \mathbb{R} \,\middle|\, \omega \neq \frac{\pi}{\tau} n, \quad P(\sigma + i\omega) \neq 0 \right\}, \tag{16}$$

where $n \in \mathbb{Z}$. Furthermore, it has a single root in $s = \sigma$ iff $P(\sigma) \neq 0$ and:

$$k_\delta = \frac{\tau}{e^{-\tau\sigma} - 1} \left(k_p + G^{-1}(\sigma) \right), \quad \text{for} \quad \sigma \neq 0, \tag{17}$$

$$k_p = -\frac{q_0}{p_0}, \quad k_\delta \in \mathbb{R}, \quad \text{for} \quad \sigma = 0. \tag{18}$$

4.3 PδI Controller

Consider the use of the $P\delta I$ controller shown in (4). The control law corresponding to this scheme can be described by:

$$u(t) = k_p e(t) + k_i \int_0^t e(v - \tau) dv. \tag{19}$$

Notice that this is basically a classical PI controller in which the error signal is delayed by finite constant amount of time τ before integrating it. As mentioned before, the main reason for adding this delayed action to this controller is to study the behavior of the closed-loop response as τ is varied. In other words, to have an extra degree of freedom in the tuning of a PI-alike controller. In order to study its stability, the characteristic equation of the closed-loop system rewrites as $C_i(s)G(s) + 1 = 0$, which leads to:

$$\Delta_i(s) = s\left(k_p + G^{-1}(s)\right) + k_i e^{-\tau s}. \tag{20}$$

The following result summarized in this section works as tools for describing the behavior of the roots of this equation.

Proposition 2. Let $\tau \in \mathbb{R}_+$ and $\sigma \in \mathbb{R}$ be fixed values. Then, the characteristic equation (20) has at least one root in $s = \sigma + i\omega$, iff:

$$k_p = -\Re(\sigma,\omega) + \frac{\omega \sin(\tau\omega) - \sigma \cos(\tau\omega)}{\sigma \sin(\tau\omega) + \omega \cos(\tau\omega)} \Im(\sigma,\omega), \quad (21)$$

$$k_i = \frac{\sigma^2 + \omega^2}{\sigma \sin(\tau\omega) + \omega \cos(\tau\omega)} \Im(\sigma,\omega) e^{\tau\sigma}, \quad (22)$$

with $\omega \in \Omega_i$, where the set Ω_i is defined by:

$$\Omega_i := \{\omega \in \mathbb{R}\,|\,\omega \cot(\tau\omega) + \sigma \neq 0, P(\sigma + i\omega) \neq 0\}, \quad (23)$$

where $n \in \mathbb{Z}$. Furthermore, it has a single root in $s = \sigma$ iff $P(\sigma) \neq 0$ and:

$$k_i = -\sigma\left(k_p + G^{-1}(\sigma)\right) e^{\tau\sigma}. \quad (24)$$

Furthermore, we present an additional proposition for computing the stabilizing interval of the delay value given a stabilizing triplet (k_p, k_i, τ).

Proposition 3. Let (k_p, k_i, τ^*) be a stabilizing triplet, then, the closed-loop system is asymptotically stable for any delay value $\tau \in [\tau^*, \tau_c)$, where:

$$\tau_c = \min\{\tau \in \mathbb{R}\,|\,\tau(\omega^*) > 0, \omega^* \in \Omega_p\}, \quad (25)$$

in which $\tau(\omega^*)$ is computed as:

$$\tau(\omega^*) = \frac{1}{\omega^*}\left[\arg\left\{\frac{k_i P(i\omega^*)}{i\omega^*(k_p P(i\omega^*) + Q(i\omega^*))}\right\} + (2n+1)\pi\right], \quad (26)$$

for $n \in \mathbb{Z}$ and where the set Ω_p is defined as the set of all real roots of the following equation:

$$|k_i P(i\omega^*)|^2 - {\omega^*}^2 |k_p P(i\omega^*) + Q(i\omega^*)|^2 = 0. \quad (27)$$

5. ILLUSTRATIVE EXAMPLES

In this section, we describe in detail how the methodology explained in Section 4.1 can be applied for two different examples of second-order systems using the $P\delta$ and $P\delta I$ controllers.

Example 3. Consider the following transfer function:

$$G(s) = \frac{1}{s^2 - 3s + 5}, \quad (28)$$

which two poles lie on $s = 1.5 \pm 1.65i$. Since it has two roots on the RHP, it is an unstable open-loop system. Now, let us consider the application of a PD controller, the characteristic equation of the closed-loop system can be computed as:

$$s^2 + (kd - 3)s + k_p + 5 = 0. \quad (29)$$

By Hurwitz criterion, it is easy to observe that in order to achieve closed-loop stability the application of a derivative action is mandatory so every coefficient has the same sign. This is the case of a simple PD controller with $k_p > -5$ and $k_d > 3$. To avoid such a derivative action we propose the use of the $P\delta$ controller in the following lines.

Considering the open-loop transfer function (28) and the $P\delta$ controller shown in (3) the characteristic equation of the closed-loop system can be computed as:

$$\Delta_\delta(s) = s^2 - 3s + 5 + k_p + k_\delta\left(\frac{1 - e^{-\tau s}}{\tau}\right) = 0. \quad (30)$$

Using Proposition 1 with a fixed delay value $\tau = 0.04s$ we compute the *stability crossing curves* $(\mathcal{T}(0))$ as some curves from the set (σ) with $\sigma > 0$. These graphical results are shown in Figure 2 on the $k_p - k_\delta$ parameters-space. We use the curves from the set $\mathcal{T}(\sigma)$ with $\sigma > 0$ for discriminating the unstable regions to further find a *stability region*.

Finally, in order to test this result we choose three different controllers, being c_1 and c_2 stable and c_3 unstable controllers as is depicted in Fig. 2. Some simulation results using these controller parameters are shown in Fig. 3. Also in this figure, we show the closed-loop response of a PD controller using the same gains as the $P\delta$ controller $((k_p, k_d) = (k_p, k_\delta))$. As expected, these results corroborate the graphical results on figure 2.

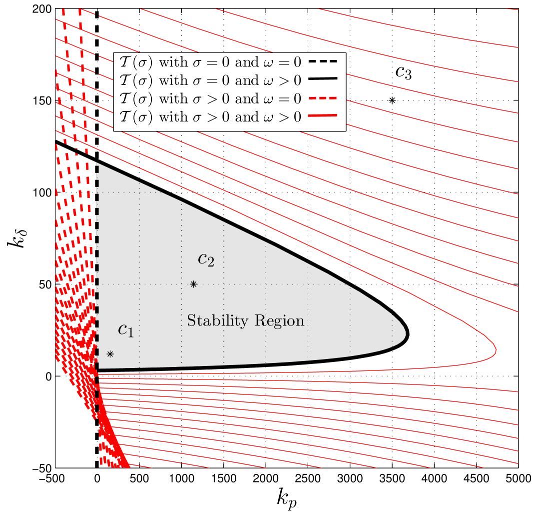

Fig. 2. Stability analysis in the $k_p - k_\delta$ parameters space

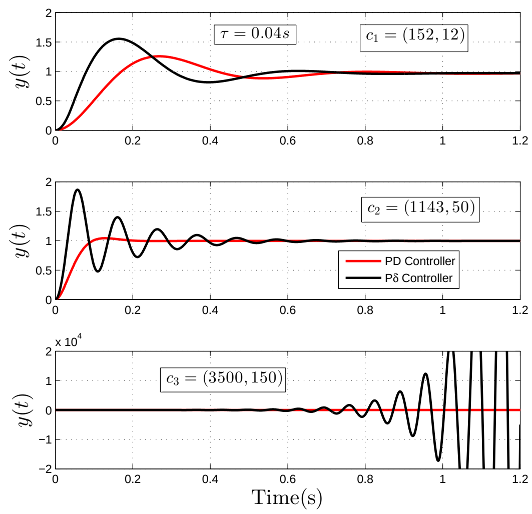

Fig. 3. Closed-loop system response comparison between a PD controller and a $P\delta$ controller

Example 4. Consider the following open-loop transfer function:

$$G(s) = \frac{1}{s^2 + 2s + 3}, \quad (31)$$

with poles $s_{1,2} = -1 \pm 1.41i$. In this stable open-loop system we are considering the problem of a controller design such that the steady state error is equal to zero. This can be easily achieved by a simple PI controller, however, as stated before we aim to use the delayed action to manipulate the closed-loop response.

Considering the open-loop transfer function (31) and the $PI\delta$ controller shown in (4) the characteristic equation of the closed-loop system can be computed as:

$$\Delta_\delta(s) = s^3 + 2s^2 + (3 + k_p)s + k_i e^{-\tau s} = 0. \quad (32)$$

Following the same methodology explained in the last example using a fixed delay $\tau = 0.5s$ we find a stability region as shown in Fig. 4. In a similar way, we test its reliance with three different controllers, c_4 and c_5 stable controllers and c_6 an unstable one. We show some simulation results presented in Fig. 5 which corroborates this result in comparison to a simple PI controller $\tau = 0$. From this comparison, we can notice the damping added with controller c_5 relative to the controller c_4 and also to the simple PI controller. At last, we show another

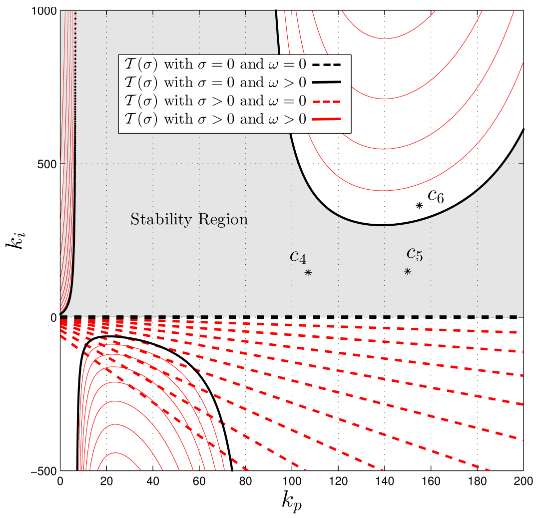

Fig. 4. Stability analysis in the $k_p - k_i$ parameters space

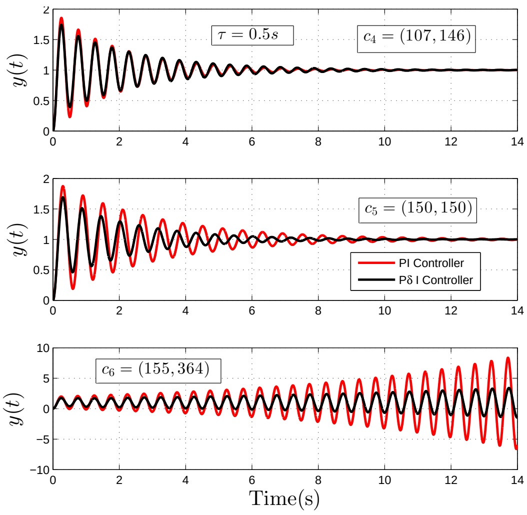

Fig. 5. Closed-loop system response comparison between a PI controller and a $PI\delta$ controller

way to tune the delayed action in the $P\delta I$ controller. Consider the $P\delta I$ gains $(k_p, k_i) = (5, 5)$ with $\tau = 0$, the roots of the characteristic equation are $s_{1,2} = -0.64 \pm 2.58i$ and $s_3 = -0.7$. Since this is a stable system, we consider Proposition 3 to compute the stabilizing delay interval $\tau = (0, \tau_c)$, obtaining $\tau_c = 2.15s$. Some simulation results depicting the continuous variation of the closed-loop response as τ is varied on this interval are presented in Fig. 6. In this figure, one can notice how from $\tau = 0$ to $\tau = 0.2\tau_c$ we are able to inject damping to the closed-loop response. Recall that this task is commonly achieved by the use of a derivative action which we are avoiding. Also in this figure, we show the behavior of the closed-loop response as the other part of the interval is considered, leading as expected to instability.

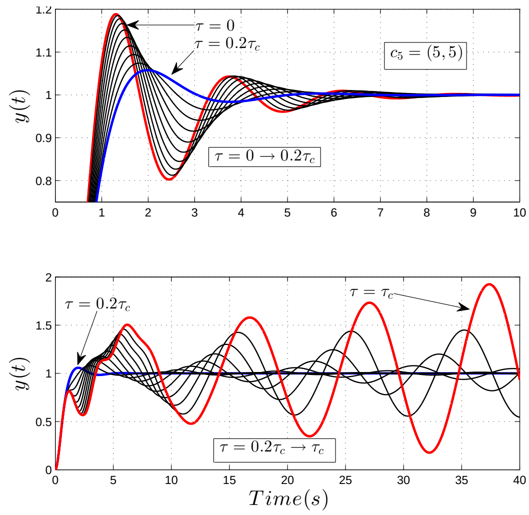

Fig. 6. Closed-loop system response using a $P\delta I$ controller under different time delays with $\tau \in (0, \tau_c)$

5.1 PD Alike Controllers Comparison

As mentioned in the Introduction, non-desired high frequency sensors noise can potentially be amplified by the use of a classical PD controller. Even some filtered schemes have been proposed in the literature (see, for instance O'Dwyer (2009)) to circumvent such scenario. Probably the most direct example of this is the PD_f controller (PD controller with filtered derivative) shown below:

$$C_f(s) := K_c \left[1 + \frac{T_d}{1 + \frac{T_d}{N} s} \right], \tag{33}$$

where K_c, T_d and N are real parameters and can be described as a PD controller with a low pas filter in the derivative action with break frequency $w_o = \frac{N}{T_d}$. In this section, we show an example in which we compare three different schemes using the PD, PD_f and $P\delta$ controllers applied to the open-loop system shown in (31). Also, we evaluate these considering a tracking problem and under high-frequency noise disturbances due to sensors noise.

Example 5. Consider a low frequency reference signal $r(t) = sin(2\pi f_R)$ with $f_R = 1Hz$ and a high frequency noise signal in the error as $e(t) = r(t) - [y(t) + n(t)]$, where $n(t) = sin(2\pi f_n)$ with $f_n = 50Hz$. Furthermore, in order to make an equivalent comparison regarding controllers tuning we focus on the proportional and derivative gains analogies inside each topology. That is, using the $P\delta$ controller gains $c_7 = (700, 80)$ with $\tau = 0.04s$, this translates as $(k_p, k_d) = (k_p, k_\delta)$ for the PD controller and $(K_c, T_d) = (k_p, \frac{k_\delta}{k_p})$ for the PD_f controller. Finally, for this last we choose $N = T_d \omega_0$ for achieving a break frequency of $w_o = 2\pi f_o$ with $f_o = 40Hz$ (below the noise signal frequency). The results of this tests are shown in Fig 7.

Now, with the purpose of making a quantitative comparison we propose the following performance indicators: e_A-Amplitude of the ripple in the error signal in steady state due to noisy behavior, e_M-Maximum peak of the absolute value of the error signal and t_s-Settling time. All of this indicators values are shown in Tab. 1. Using this

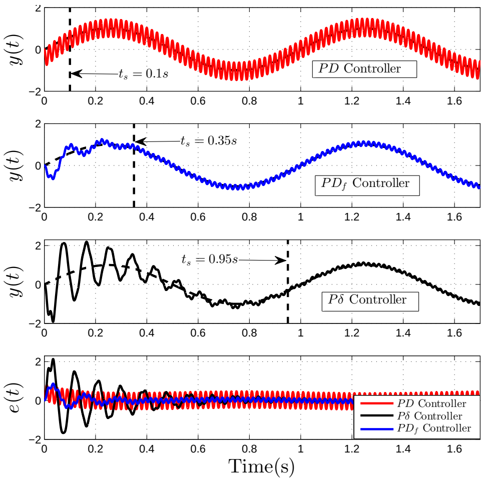

Fig. 7. Tracking performance and noise rejection comparison between the PD, PD_f and $P\delta$ controllers.

Table 1. Performance indicators values

Controller	e_A	e_M	t_s	A
PD	0.44	0.802	0.1	0.225
PD_f	0.085	0.870	0.35	0.129
$P\delta$	0.038	2.137	0.95	0.507

information we show in Fig. 8 a radar chart normalized with respect to the worst case for a given indicator. Also, another interesting indicator of this figure is the areas A of the polygons depicted in this figure for each controller. The ideal scenario concerns to the case in which this area is zero, that is no error signal. In other words, as this area is minimized, a controller fulfills more suitably the ideal indicators.

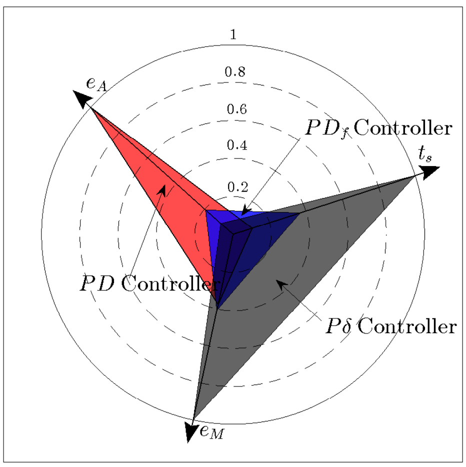

Fig. 8. Radar chart comparison between the PD, PD_f and $P\delta$ controllers for tracking performance and noise rejection.

Finally, from the analysis depicted in Fig. 8 and Tab 1 we can notice that the controller better fulfilling this indicators is the PD_f controller. However, even though the $P\delta$ controller has the bigger area in this radar chart it is worth noticing that is the one that achieves the better high frequency noise rejection without implementing any additional behavior.

6. CONCLUDING REMARKS

A methodology for the design of the $P\delta I$ and $P\delta$ controllers such that the closed-loop system is asymptotically stable and some motivating examples of the use of those is presented. We like to enhance the fact that this methodology avoids any crossing direction analysis and is presented as practical guidelines to develop a simple control scheme design. Moreover, simulation results corroborating these ideas are shown to enhance the advantages of using these delayed controllers. Likewise, a PD controllers alike comparison considering a tracking problematic under high-frequency noise is addressed

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